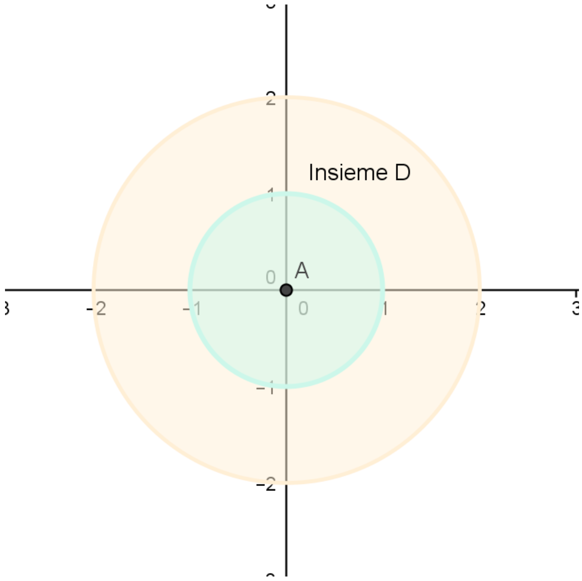
Esercizio 7:

Calcolare l'integrale:

$$\iint_D x^2(4+x^2y)\,dxdy$$

Dove D è la corona circolare di centro l'origine e raggi 1 e 2.

Svolgimento:



Uso le coordinate polari:

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \end{cases}$$

Trovo lo Jacobiano:

$$\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} cos\theta & -\rho sin\theta \\ sin\theta & \rho cos\theta \end{vmatrix} = \rho cos^2\theta + \rho sin^2\theta = \rho(cos^2\theta + sin^2\theta) = \rho$$

Estremi di interazione:

$$0 \le \theta \le 2\pi \quad 1 \le \rho \le 2$$

Funzione integranda:

$$\varphi(\theta,\rho) = \rho^2 cos^2\theta(4+\rho^2 cos^2\theta \rho sin\theta)$$

$$\iint_D x^2(4+x^2y)\,dxdy = \int_0^{2\pi}\int_1^2 \rho^2 cos^2\theta(4+\rho^3 cos^2\theta sin\theta)\,\rho d\rho d\theta =$$

$$= \int_0^{2\pi}\int_1^2 (4\rho^3 cos^2\theta + \rho^6 cos^4\theta sin\theta)\,d\rho d\theta = \int_0^{2\pi}\left[\int_1^2 (4\rho^3 cos^2\theta + \rho^6 cos^4\theta sin\theta)\,d\rho\right] d\theta =$$

$$= \int_0^{2\pi}\left[\frac{4\rho^4}{4} cos^2\theta + \frac{\rho^7}{7} cos^4\theta sin\theta \Big|_1^2\right] d\theta =$$

$$= \int_0^{2\pi} \left[2^4 cos^2\theta + \frac{2^7}{7} cos^4\theta sin\theta - \left(1^4 cos^2\theta + \frac{1^7}{7} cos^4\theta sin\theta \right) \right] d\theta =$$

$$= \int_0^{2\pi} \left(16cos^2\theta + \frac{128}{7} cos^4\theta sin\theta - cos^2\theta - \frac{1}{7} cos^4\theta sin\theta \right) d\theta =$$

$$\int_0^{2\pi} \left(15cos^2\theta + \frac{127}{7} cos^4\theta sin\theta \right) d\theta =$$

$$= 15 \int_0^{2\pi} cos^2\theta d\theta + \frac{127}{7} \int_0^{2\pi} cos^4\theta sin\theta d\theta$$

Calcolo il primo integrale:

$$\int cos^2\theta d\theta =$$

Per parti:

$u = cos\theta \quad du = -sin\theta d\theta$

$dv = cos\theta d\theta \quad v = sin\theta$

$$= cos\theta sin\theta + \int sin^2\theta d\theta = cos\theta sin\theta + \int (1 - cos^2\theta) \, d\theta$$

$$\int cos^2\theta = cos\theta sin\theta + \theta - \int cos^2\theta d\theta$$

$$2 \int cos^2\theta d\theta = cos\theta sin\theta + \theta$$

$$\int cos^2\theta = \frac{cos\theta}{2} + \frac{\theta}{2} + C$$

Con C costante arbitraria. Considerando il coefficiente e gli estremi di integrazione si ha:

$$15 \int_0^{2\pi} cos^2\theta d\theta = 15 \left[\frac{cos\theta}{2} + \frac{\theta}{2} \Bigg|_0^{2\pi} \right] = 15 \left[\frac{cos(2\pi)}{2} + \frac{2\pi}{2} - \left(\frac{cos0}{2} + \frac{0}{2} \right) \right] =$$

$$= 15 \left(\frac{1}{2} + \pi - \frac{1}{2} \right) = 15\pi$$

Calcolo il secondo integrale:

$$\int cos^4\theta sin\theta d\theta =$$

Per parti:

$u = cos^4\theta \quad du = -4cos^3\theta sin\theta d\theta$

$dv = sin\theta d\theta \quad v = -cos\theta$

$$= -cos^5\theta - 4 \int cos^4\theta sin\theta d\theta$$

$$\int cos^4\theta sin\theta d\theta = -cos^5\theta - 4\int cos^4\theta sin\theta d\theta$$

$$5\int cos^4\theta sin\theta d\theta = -cos^5\theta$$

$$\int cos^4\theta sin\theta d\theta = -\frac{cos^5\theta}{5} + C$$

Con C costante arbitraria. Considerando il coefficiente e gli estremi di integrazione si trova:

$$\frac{127}{7}\int_0^{2\pi} cos^4\theta sin\theta d\theta = \frac{127}{7}\left[-\frac{cos^5\theta}{5}\Big|_0^{2\pi}\right] = \frac{127}{7}\left[-\frac{cos^5(2\pi)}{5} - \left(-\frac{cos^5 0}{5}\right)\right] =$$

$$= \frac{127}{7}\left[-\frac{1}{5} - \left(-\frac{1}{5}\right)\right] = \frac{127}{7}\left(-\frac{1}{5} + \frac{1}{5}\right) = 0$$

Quindi:

$$\iint_D x^2(4 + x^2y)\,dxdy = \int_0^{2\pi}\int_1^2 \rho^2 cos^2\theta(4 + \rho^3 cos^2\theta sin\theta)\,\rho d\rho d\theta = 15\pi$$