Esercizio 2

Calcolare:

$$\lim_{n\to+\infty}\left\{[n^2+\ln(n!)+\cos n]\left[\sin\left(\frac{1}{n}\right)\ln(n+1)-arctg\left(\frac{1}{n}\right)\ln(n-1)\right]\right\}$$

Svolgimento

Raccogliamo n^2:

$$\lim_{n\to+\infty}\left\{n^2\left[1+\frac{\ln(n!)}{n^2}+\frac{\cos n}{n^2}\right]\left[\sin\left(\frac{1}{n}\right)\ln(n+1)-arctg\left(\frac{1}{n}\right)\ln(n-1)\right]\right\}=$$

Ma:

$$\lim_{n\to+\infty}\frac{\ln(n!)}{n^2}=0 \quad e \quad \lim_{n\to+\infty}\frac{\cos n}{n^2}=0$$

Sostituendo:

$$=\lim_{n\to+\infty}\left\{n^2\left[\sin\left(\frac{1}{n}\right)\ln(n+1)-arctg\left(\frac{1}{n}\right)\ln(n-1)\right]\right\}=$$

$$=\lim_{n\to+\infty}\frac{\sin\left(\frac{1}{n}\right)\ln(n+1)-arctg\left(\frac{1}{n}\right)\ln(n-1)}{\frac{1}{n^2}}=$$

$$=\lim_{n\to+\infty}\left[\frac{\sin\left(\frac{1}{n}\right)\ln(n+1)}{\frac{1}{n^2}}-\frac{arctg\left(\frac{1}{n}\right)\ln(n-1)}{\frac{1}{n^2}}\right]=$$

$$=\lim_{n\to+\infty}\left[\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\frac{\ln(n+1)}{\frac{1}{n}}-\frac{arctg\left(\frac{1}{n}\right)}{\frac{1}{n}}\cdot\frac{\ln(n-1)}{\frac{1}{n}}\right]=$$

Ma:

$$\lim_{n\to+\infty}\frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}=1 \quad e \quad \lim_{n\to+\infty}\frac{arctg\left(\frac{1}{n}\right)}{\frac{1}{n}}=1$$

Sostituendo:

$$=\lim_{n\to+\infty}\left[\frac{\ln(n+1)}{\frac{1}{n}}-\frac{\ln(n-1)}{\frac{1}{n}}\right]=\lim_{n\to+\infty}[n\ln(n+1)-n\ln n-1]=$$

$$=\lim_{n\to+\infty}[\ln(n+1)^n-\ln(n-1)^n]=\lim_{n\to+\infty}\left[\ln\left(\frac{n+1}{n-1}\right)^n\right]=$$

Voglio arrivare a scrivere la funzione di cui devo calcolare il limite nella forma $\left(1+\frac{1}{f(n)}\right)^{f(n)}$.
Sottraggo e aggiungo 1 al numeratore:

$$= \lim_{n\to+\infty}\left[\ln\left(\frac{n+1-1+1}{n-1}\right)^n\right] = \lim_{n\to+\infty}\left[\ln\left(\frac{n-1}{n-1}+\frac{2}{n-1}\right)^n\right] =$$

$$= \lim_{n\to+\infty}\left[\ln\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^n\right] =$$

Divido e moltiplico l'esponente per 2:

$$= \lim_{n\to+\infty}\left[\ln\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{n}{2}\cdot 2}\right] = \lim_{n\to+\infty}\left[2\ln\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{n}{2}}\right] =$$

$$= \lim_{n\to+\infty}\left\{2\ln\left[\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{n}{2}}\cdot\frac{\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{1}{2}}}{\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{1}{2}}}\right]\right\} =$$

$$= 2\lim_{n\to+\infty}\ln\left[\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{n}{2}-\frac{1}{2}}\cdot\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{1}{2}}\right] =$$

$$= 2\lim_{n\to+\infty}\left[\ln\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)^{\frac{1}{2}(n-1)}+\frac{1}{2}\ln\left(1+\frac{1}{\frac{1}{2}(n-1)}\right)\right] =$$

Ricordando che:

$$\lim_{n\to+\infty}\left(1+\frac{1}{f(n)}\right)^{f(n)} = e \ se \lim_{n\to+\infty} f(n) = \pm\infty$$

$$= 2\ln e = 2$$

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